Educ Stud Math (2011) 77:313–329
DOI 10.1007/s10649-011-9302-9

Evolution, learning, and semiotics from a Peircean point of view

Michael Friedrich Otte

Published online: 11 February 2011

Abstract One of the most salient arguments in favor of a semiotic approach, put forward on various occasions among others by Luis Radford, claims that semiotics is most appropriate to treat the interaction between socio-cultural and objective aspects of knowledge problems. But if we want to take such claims seriously, we have to undertake revisions of our basic conceptions about reality, existence, cognition, and cultural development. The semiotic evolutionary realism of Charles S. Peirce provides, or so it appears, an appropriate basis to such intentions. Man is a sign, Peirce had famously said, and "thought is more without us than within. It is we that are in it, rather than it in any of us" (Peirce CP 8.256). And as there is no thought without a sign, we have to accept thoughts, concepts, theories, or works of art as realities *sui generis*. Concepts or theories have to be recognized as real before we can ask for their meaning or relevance. This was the problem that concerned critics and protagonists of the *New Math Reform* of the 1960s and 1970s of the twentieth century, like Thom or Bruner.

Keywords Semiosis · Peirce · Continuity · Relational thinking · Metaphor · Mathematics education

1 A problem, a thesis, and a first presentation of a semiotic approach

Luis Radford has on various occasions and by a number of arguments substantiated the claim of a fundamental importance of a semiotic approach to mathematics education (see for example 2006, p. 7ff.). One of the most salient arguments indicated that the semiotic approach is most appropriate to treat the interaction between socio-cultural and objective aspects of knowledge and knowing.

Of course, school does contain a lot of problems and many of them have to be solved by psychological, organizational, social, or political means. But there are also problems that

M. F. Otte (✉)
Institut fur Didaktik der Mathematik, University of Bielefeld Universitaetsstrasse,
33501 Bielefeld, Germany
e-mail: Michaelontra@aol.com

have to be approached from the point of view of mathematical practice and the dynamics of mathematics, and among them the problem of generalization—understood in social as well as in objective terms—is the most important one.

We shall in the following concentrate ourselves on this particular question and treat it from various perspectives.

In fact, these kinds of epistemic questions were responsible in the first place for the failures and achievements of the *New Math Reform* of the 1960s and 1970s of the twentieth century. The reform had gloated at the affirmation that for the first time in the history of mathematical education, one was to succeed in reconciling "intuition and logic" (G. Papy 1967, oral presentation at the *Duesseldorf Academy of Sciences*) or fundamental theoretical ideas and concrete meanings by means of the set theoretical metaphor. And to try and provide mathematical cognition with foundations *sui generis* was really a courageous undertaking. Even critics of the New Math, like R. Thom, pointed in the same direction indicating the importance of a graspable mathematical ontology. Thom in his invited lecture to the Second International Congress on Math. Education in Exeter in 1972 had said: "The real problem which confronts mathematics teaching is not that of rigor, but the problem of the development of meaning, of the existence of mathematical objects". And Jerome Bruner (1969), one of the most important protagonists of the reform, asks in a similar vein, "What do we say to a young child, asking if concepts like force or pressure really exist?" (p.54).

We understand, given the historical facts of the New Math reform, these pronouncements of Bruner and Thom as an indication of the insufficiency of set-theoretical mathematics. And we want, in fact, to present the thesis that only a semiotic approach can help, not only because it perceives the general, both in its social as well as content-related aspects or references—but also because it suggests alternatives to set-theoretical foundationalism.

A sign has meaning, but does not exist like a concrete thing, because it is a general, a type, not a token. A thing in contrast exists, but has per se no meaning at all, because for something to have a meaning it must be related to something else. Things or objects are recognized in terms of identity and difference, like in mathematical set theory; whereas representations or meanings have no perfect identity, but are related to each other in terms of likeness, analogy, or metaphor. A metaphor can be broken up into a series of relations of "equality", that are, however, not transitive—A may be near or similar to B and B to C without C belonging to the neighborhood of A—says Aristotle, such that a bold metaphor may surprise us and stimulate us to find the "common genus" (Aristotle, 1926, Rhetoric III, 10; 1410b). And Peirce expresses similar views writing: "I desire to point out that it is by taking advantage of the idea of continuity, or the passage from one form to another by insensible degrees, that the naturalist builds his conceptions" (Peirce, 1958, CP 2.646). This breaking down of differences thereby making keen metaphors more accessible to the interpreter or learner and thus developing a logic of abductive reasoning seems as fundamental indeed as the drawing of clear distinctions.

A sign is general in comparison with its objects, but it is not like a definition or a mere abstraction or a predicative general. The generality of a sign consists in the continuum of all its possible interpretations. And the interpretation of a sign consists just in the construction of another sign. We often realize this when trying to translate some text from one language into another. As Roman Jakobson, characterizing Peirce's thinking, once said: "One of the most felicitous, brilliant ideas which general linguistics and semiotics gained from the American thinker is his definition of meaning *as the translation of one sign into another system of signs* (CP 4.127)" (Jakobson, 1985, p. 251).

A sign in the sense of Peirce is a continuum because there is no definite and ultimate or absolute interpretation. And a continuum is not a collection of points, is no set of distinct existents. It is of the nature of a relationship and its parts are continua again. Already Aristotle's proposed solution for Zeno's paradoxes involves distinguishing between what he termed a "continuous" line and a line divided into parts. A unit of space can be mentally divided into ever smaller units while remaining spatially the same. As a general is, however, a more or less indeterminate and implicit entity it cannot be further developed and made concrete independently from the particular and determinate.

Let us note that Aristotle is most often regarded as the great representative of a logic and mathematics, which rests on the assumption of the possibility of clear divisions and rigorous classification.

> But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language [...] (Lovejoy, 1964, p.58)

Aristotle thereby became responsible for the introduction of the principle of continuity into natural history and since then we witness the importance of this contrast between the principle of identity, which rules logic and application and the principle of continuity so important to the development of mind. Ever since, these two principles have remained opposed to one another and even the process of mathematical knowledge has a complementary structure, reproduced nowadays in the duality of the hypothetic–deductive approach versus set theoretical foundations (mathematics as modal logic vs. mathematics as set theory).

Peirce tries to capture the structure of our possible experience by three fundamental categories, which he calls, in order to avoid premature reification, by completely abstract names, Firstness, Secondness, and Thirdness "Firstness is the mode of being of that which is such as it is, positively and without reference to anything else. Secondness is the mode of being of that which is such as it is, with respect to a second but regardless of any third". From the cognitive viewpoint, one might identify Firstness with imagination, Secondness with objective constraint, and Thirdness as semiotic mediation, as "the mode of being of that which is such as it is, in 'bringing a second and first into relation to each other'" (Peirce, 1958, CP 8.328). And on a different occasion Peirce writes: "Thirdness is the triadic relation existing between a sign, its object and the interpreting thought,...considered as constituting the mode of being of a sign" (Peirce, CP 8.332).

The essential aspect, which was pointed out by Jakobson's remark already, is that Thirdness qualifies Firstness and Secondness as dependent elements. It is for example communication as a sign process which brings people into contact with each other, rather than people being together and then deciding to make use of communication. The persons are parts of an ongoing process of interpretation and communication that rises above the subjective. "Man is a sign", Peirce had famously said, and "thought is more without us than within. It is we that are in it, rather than it in any of us" (Peirce, CP 8.256).

And it might thus be useful to quote another Peircean description of this semiotic evolution to which Jakobson referred:

> The object of a representation can be nothing but a representation of which the first representation is the interpretant...The meaning of a representation [that is, the idea which it conveys; my insertion M.O.] can be nothing but a representation. In fact it is nothing but the representation itself stripped of irrelevant clothing. But this clothing

> never can be completely stripped off; …So there is an infinite regression here. Finally the interpretant is nothing but another representation to which the torch of truth is handed along; as a representation it has its interpretant again. Lo, another infinite series. (Peirce, 1958 CP 1.339)

But there are limits of these infinities in terms of activity and communicative conventions, because a sign has to function as sign within a universe of discourse and action.

What we intend to do in the following parts of this paper is trying to develop an idea of this notion of Thirdness, as a way to better come to terms with our central problem, the problem of generalization, mentioned already.

2 The double nature of meaning as generalization and as verification

So reality is composed of distinct existent things, on the one hand, and signs or meanings, which are real in the sense of opening up possibilities and thus have to be grasped out of a dynamical and developmental perspective, on the other hand. Universals are therefore better not understood as abstractions in the common Platonic sense, but are to be conceived of as relations, because whenever we are led to recognize the relations as more important or prior to the relata we might also realize that the relational organization in question is the result of sign processes.

All objects are simply existing things without any meaning. Therefore, everything which is real in an intelligible sense is a relation or a continuum. There might be, for example, good individuals and evil ones, but in order to understand what this means one must take into account the relation or the continuum between these extremes. The only productive way to think of distinct existents is perceiving them in a relation. Only relations can be objectively grasped and communicated.

Relational thinking is particularly important in mathematics. Positive and negative numbers make sense from a relational point of view only and in order to justify the rules of calculation for negative, fractional or imaginary numbers one has to represent them in relational terms: $3=5+x$; $7x=3$; $5y=1$, etc. And when a mathematician refers to the “fundamental theorem of algebra”, he does not mean to say that the roots of an algebraic equation exist according to the common meaning of the word “existence”. In geometry, the relational perspective begins by introducing the continuity principle and later on with vector calculus. The complex number system is a vectorial system, which in addition shows the intimate connection between the continuum and the relational point of view.

School mathematics as a rule thinks differently. School mathematics is algebra but does not conceive of algebra in relational terms, considering it rather as generalized arithmetic. School algebra does not have real variables and school mathematics identifies meaning with verification and is not interested in generalization. There have also been less “anti-didactic” approaches to algebraic thinking, however. Davidov, for example, always had emphasized the timely and proper transition of the children to the ability to orient themselves in the relations of quantities and figures themselves (in the abstract relations proper) as essential condition of the children’s understanding of mathematical thought and he therefore proposed to begin mathematical teaching with an introduction of these relations themselves (see, for instance, Davidov, 1977).

But consider the following very simple situation. A teacher takes two pieces, a red and a blue one and in addition a number of weights of 1 and 2 kg, respectively, and then shows his pupils, that both pieces together weigh 2 kg and that the red one alone weighs 1 kg plus the blue one (the whole classroom episode is given in Steinbring, 2005, p. 96ff).

The pupils should now experiment with various relations of weights until they eventually find out that two blue pieces are equivalent to 14 kg. But with nothing else available, no numbers, no arithmetic, they cannot do much more. The semantics at hand is too poor and restricted. Russell had frequently criticized the axiomatic introduction of number in the sense of Peano and Hilbert on similar accounts. Russell deemed the axiomatic method to be incomplete, as unspecified terms occur within the axioms. These uninterpreted terms must indeed be specified in a way that permits one to establish a connection to the intended application. An absolute or ultimate interpretation of mathematical concepts, however, is generally neither possible nor desirable and set theoretical interpretation would not solve the problem of application. As Lebesgue frequently said: "L'arithmetique s'appliqué quand elle s'applique" ("Arithmetics applies when it applies"; for a more extended discussion see, Otte, 2003).

If we accept that the subject matter of mathematics is established by relations of equality and difference we might believe that algebra is the c onceptual part of mathematics whereas arithmetic gives the concrete meanings. Arithmetic is the richest semantics possible and the arithmetical view believes that all mathematical facts should be given in arithmetical terms. The axiomatic method of Hilbert and Noether, in contrast, is nothing but the "highest point" of relational thinking in mathematics and it began to enter first year university classrooms in Germany some years after World War II. Axiomatic mathematics or mathematics as diagrammatic reasoning represents a genetic perspective aiming at generalization. It transforms mathematics into a reality *sui generis,* like a kind of fiction, and it requires a high degree of cognitive maturity in its application.

Semiotics teaches us that one must always start from the means of activity and begin to think in corresponding terms. Affirming a mathematical fact, like, the angle sum in a triangle amounts to two right angles, for example, becomes meaningful from a mathematical point of view as soon as we are able to discourse on it in theoretical terms or conceive of a relevant proof. As a fact in itself it is quite improbable and can neither exactly be verified even by the most precise measurements, nor communicated. The affirmation of a fact becomes therefore transformed into a hypothetic conditional statement. For instance: "If Euclid's parallel postulate is assumed or valid, then the sum of angles in a triangle is 180°". This view led to Hilbert's "implicit definitions" and to modern axiomatics.

Mathematics is, as Peirce defines it, "the science which draws necessary conclusions" (Peirce, 1958, CP 3.558). And being "the science which draws necessary conclusions, it busies itself with mere hypotheses about ideal states of things" (Peirce, CP 3.558), rather than with particular existents. "It is the study of pure hypothesis regardless of any analogies they may have in our universe" (1982, NEM 4:149, 1958, CP 3.560). And further: "Mathematics is purely hypothetical: it produces nothing but conditional propositions" (CP 4.240). It also follows that theories become realities *sui generis* in relation to concrete reality. This means that they cannot simultaneously be theories of their own application.

If we take Boole or Peirce and later Peano or Hilbert seriously, we must accept that only the pair of syntactical structure and intended applications together provides a complete concept of mathematics. The foundation of mathematics cannot be separated from its applications. But semantic questions cannot be answered by set-theoretical universalism alone. Set theoretical ontology is no substitute for context related analysis and interpretation. Thinking is perception, is taking a certain perspective seeing an A as a B: A=B, but knowledge depends on reasoning from such premises.

To the question posed by students and philosophers alike "what are numbers?" we can thus answer either in terms of model theoretical semantics—the favorite model being the straight line conceived of as a set of points—or we start from a prototypical or metaphorical notion of concepts and rely on the idea of knowledge as reasoning and of meaning as use. In both cases, one has to come to terms; however, with the fact that mathematical truth does not mean truth according to the traditional objectivist understanding.

> Davidov justified his approach saying that the traditional arithmetic teaching is trying to do away as quickly as possible with the introduction of numbers, without revealing the objective content of the number concept. This is the result of a separation of the teaching of mathematical concepts from their genesis, a result of the empiricist theory of generalization. (Otte, 1976, pp. 481–482)

But his critics, like Zankov, have argued that Davidov ignores the "mathematical experience" of the children and the familiarity with numbers which they bring with them when they come to school (Otte, 1976, p. 485ff.).

A sign is no sign if it does not function as such. Nevertheless, no sign or meaning is to be identified with any number of its applications or uses. It is a universal, which, however, cannot be separated from its concrete applications or contextual interpretations. Signs are generals but they function as signs only through their concrete expressions. This led some people to the conclusion that all meaning is subjective and thus contingent.

So the central problem of mathematics education is the question of how general and particular are related, in socio-cultural context, as well as with respect to the problem of the objectivity of abstract theoretical knowledge.

3 History of science and the subject–object relation

Classical science is the science of the general or universal. According to the Aristotelian model of science, the subject matter of knowledge is substances and their essential characteristics. There cannot be a science of the particular because the individual or particular is always insecure and contingent. To perceive something means to see it as essentially falling under some universal. Objective laws never determine the individual and particular.

Aristotelian science was contemplative and analytic and the development of logic was essential to its creation. Aristotle's achievement in creating this new logic and together with it science as we know it, was almost without predecessor. Aristotle's *Posterior Analytics* is the first elaborated theory in the Western philosophical and scientific traditions of the nature and structure of science and its influence reaches well into our times. It has long been accepted with such a degree of unanimity that nobody even thought of imputing special merit to Aristotle for his establishment of it.

Modern scholars despise this Aristotelian model of science and analytical philosophy and modern logic have declared the problem of generalization to be an unsolvable or pseudo-problem and have in consequence drawn an absolute distinction between theory and reality. Individualists, "existentialists", like Richard Rorty or Gregory Bateson join party here with formalists and logical objectivists, strange bed fellows indeed. They all start from the assumption of a radical separation between subject and object and then draw their individual consequences. They are all Cartesians of some sorts, but do criticize Descartes' representational theory of knowledge. But such an attitude renders everything merely contingent.

The logical empiricists believe in the formalization of language as a cure. Neo-pragmatists like Rorty do not believe in such a cure but consider human life and culture as utmostly contingent. In his famous and much discussed treatise *Philosophy and the Mirror of Nature*, Richard Rorty identifies the hypostatization of universals as the source of 'all evil':

> There would not have been thought to be a problem about the nature of reason had our race confined itself to pointing out particular states of affairs—warning of cliffs and rain, celebrating individual births and deaths. But poetry speaks of man, birth, and death as such, and mathematics prides itself on overlooking individual details. When poetry and mathematics had come to self-consciousness—when men like Ion and Theaetetus could identify themselves with their subjects—the time had come for something general to be said about knowledge of universals. Philosophy undertook to examine the difference between knowing that there were parallel mountain ranges to the west and knowing that infinitely extended parallel lines never meet, the difference between knowing that Socrates was good and knowing what goodness was. So the question arose: What are the analogies between knowing about mountains and knowing about lines, between knowing Socrates and knowing the Good? When this question was answered in terms of the distinction between the eye of the body and the Eye of the Mind, thought, intellect, insight was identified as what separates men from beasts. There was, we moderns may say with the ingratitude of hindsight, no particular reason why this ocular metaphor seized the imagination of the founders of Western thought. But it did…. (Rorty, 1980, p. 38)

And Gregory Bateson the eminent anthropologist and semiotician wrote as follows:

> We shall note elsewhere in this book that there is a deep gulf between statements about an identified individual and statements about a class. Such statements are of *different logical type,* and prediction from one to the other is always unsure…In the theory of history, Marxian philosophy…insists that the great men who have been the historic nuclei for profound social change or invention are, in a certain sense, irrelevant to the changes they precipitated. It is argued, for example, that in 1859, the occidental world was ready and ripe (perhaps overripe) to create and receive a theory of evolution that could reflect and justify the ethics of the Industrial Revolution.
>
> From that point of view, Charles Darwin himself could be made to appear unimportant. If he had not put out his theory, somebody else would have put out a similar theory within the next five years. Indeed, the parallelism between Alfred Russel Wallace's theory and that of Darwin would seem at first sight to support this view. ….
>
> But, of course, it *does* matter who starts the trend. If it had been Wallace instead of Darwin, we would have had a very different theory of evolution today. The whole cybernetics movement might have occurred 100 years earlier as a result of Wallace's comparison between the steam engine with a governor and the process of natural selection. (Bateson, 1980, pp. 46–48)

If what Bateson says were strictly true; humans would be completely lost in this world. Even their own daily actions might have completely unpredictable outcomes. And Bateson himself admits that development is unpredictable such that the "whole cybernetics movement *might* [emphasis added! M.O.] have occurred 100 years earlier" (Bateson, 1980, p. 48).

It is true, however, that the Scientific Revolution of the sixteenth/seventeenth centuries, which started within the cities of Italy and the Netherlands, was prompted by rather instrumentalist perspectives and progressed under new individualistic and pragmatic orientations. That we have to understand ourselves as parts of a more general whole, as social beings, for instance, even when exclusively interested in our own personal interests, came to be noticed much later only, during the Industrial Revolution of the nineteenth century.

But Bateson is not completely wrong. And why this is so becomes understandable from a semiotic point of view.

Firstly, with respect to any cognitive activity it seems very relevant indeed which definition is chosen, which perspective is taken, or how a problem situation is represented, exactly because the various expressions neither have the same sense nor evoke the same ideas. Two concepts A and B are not the same even if contingently or necessarily all As are Bs and vice versa, because different concepts help to establish different kinds of relationships and thus influence development in different ways. Two concepts could be extensionally equivalent and yet might function differently, within a certain cognitive context.

Secondly, the way something is represented is also of great importance with respect to the historical growth of knowledge. This is very nicely illustrated by an argument of Nobel Prize winner Richard Feynman, in which he compares the different formulations of classical mechanics as given by Newton, Lagrange, and Hamilton, respectively.

> Mathematically each of the three different formulations, Newton's law, the local field method and the minimum principle, gives exactly the same consequences. What do we do then? You will read in all the books that we cannot decide scientifically on one or the other. That is true. ... But psychologically they are different because they are completely unequivalent when you are trying to guess new laws. As long as physics is incomplete, and we are trying to understand the other laws, then the different possible formulations may give clues about what might happen in other circumstances. (Feynman, 1965, p. 53)

And David Bohm provides an additional argument in this direction:

> It was widely believed in the nineteenth century that Newtonian dynamics and Hamilton–Jacobi wave theory of dynamics were essentially the same. Nevertheless we can now see that the difference between wave dynamics and particle dynamics was potentially of very great relevance in the sense that the former can lead in a natural way to quantum theory, while the latter cannot. (Bohm in Suppe, 1974, p. 383)

Representations are neither arbitrarily chosen nor received or absorbed with equal readiness. Darwin's analogies, being firmly related to the socio-economical state and the political debates of his home country—Darwin's hatred of slavery and the intense political struggles of his day on this issue seem to have been firing his evolutionary work, for example (see Desmond & Moore, 2009)—and connected with that his profound acquaintance with the expertise of farmers and breeders, might perhaps have been more easily accessible or understandable then Wallace's. No author alone determines the meaning of his text and this fact has led people even to claim that no text "does have a literal meaning in the sense of some irreducible content that survives the sea change of situations" (Fish, 1980, p. 277; see also Otte, 1986, pp. 174–176). But even if all meaning would be metaphorical, there would be some continuity, contrary to what Fish or Rorty believe. It is the text, conceived of as code and context, which determines its meaning.

Bateson himself recognized such facts in the case of evolution theory:

> The evolution of the horse from *Eohippus* was not a one-sided adjustment to life on grassy plains. Surely the grassy plains themselves were evolved *pari passu* with the evolution of the teeth and hooves of the horses and other ungulates. Turf was the evolving response of the vegetation to the evolution of the horse. It is the *context* which evolves. (Bateson, 1973, p. 128)

Considerations of such a kind may suggest the idea that generality occurs in two forms, namely predicative generality on the one hand, and continuity on the other. Generality as continuity means that a general is something not specified in every respect and "the idea of a general involves the idea of possible variations" (Peirce, 1958, CP 5.102). Thus, if everything would be continuous with or connected to everything else and thus would have meaning, all knowledge would become analytical, as in the Aristotelian model of science. Aristotle (1926) does not recognize a *collective* as well as a *distributive* concept of a multiplicity of elements, but argues in *Physics*, that "it is impossible that something that is continuous be constituted from indivisibles, e.g., a line from points, the line being continuous but the point indivisible" (book VI, 231a26).

But not everything has a meaning, that is, there are signs and existent things in the world and the human subject is an individual, distinct from everybody else, as well as a social and cultural being. Hence, results the problem of the relationship between particular and universal.

4 The industrial revolution and new educational demands

Metaphor claims to give the essence of a thing, but gives rather this essence as being the essence of a representation of that thing and the essence of a representation is but another representation. So it is under suspicion of an infinite sequence of interpretations. Universals are signs, according to Peirce, and as such they must be applied or interpreted and for that purpose the subject has to create another sign, which becomes an interpretation of the first sign.

Analogy and metaphor became prominent as indispensable tools of imagination and communication during the nineteenth century, with its new demands on research and scientific education. Whereas individual certainty and foundational problems were the main concern of eighteenth century Enlightenment, science and mathematics became primarily preoccupied with growth and generalization, now at the beginning of the nineteenth century. Generalization became essential in order to accommodate for an ever greater variety of perspectives and contextual interests. Think for example of the diversification of geometry, since the teachings of Monge, making the nineteenth century the new golden age of geometry (we shall come back to this in the last part). And mathematics must also generalize not to become a mechanical and dead enterprise, rather than a creative activity.

The Humboldtian conception of a university, for the first time in history, put research and education (Bildung) on equal footing.

> The ultimate cause of this burgeoning of German scholarship was the new and at that point uniquely German conviction that the professor's responsibility is not only to transmit academic learning but also to expand it, through criticism and research. This ideal of the professor's proper function can be called the research imperative. (Turner, 1981, p. 109)

The essential character of mathematics is due to its power of generalization, lies in its power to create ideal objects, that is, "in the peculiar kind and degree of its abstraction" (Peirce, CP 4.231). Schelling and Hegel saw the solution to this problem of generalization in a wonderful and secret faculty: *intellectual intuition.* This reminds one of Jerome Bruner's emphasis on the "fundamental ideas" for learning and in his autobiographical essays Bruner says that this put him "quite off the main line of American education theory", because "the American educational tradition had always favored experience over reason, facts over structure, and thoroughness over intuition" (Bruner, 1983, p. 184). And in the same manner as Bruner's ideas had been welcomed during the times of the New Math Reform, which was caused by the so called "Sputnik shock," Schelling's and the other German idealists' ideas were a part of the Humboldtian reform, which occurred in reaction to the superiority of the imperial France of Napoleon. Both reforms tried to put pure mathematics on footings of its own, because both aimed at generalization, considering the individual just as representative of very general intellectual and societal aspirations.

Schelling (1775–1854) defines intellectual intuition in general terms; as "the capacity to see the universal in the particular, the infinite in the finite, and unite both in a living unity." An intellectual intuition, he explains, consists in grasping an individual as a member of a whole, in seeing how its essential nature or inner identity depends on the totality of which it is a part (Beiser, 2008, p. 580ff.).

The familiar "general triangle" of school geometry comes readily to mind here. From a semiotic perspective, a general triangle is a free variable to be treated as representative of a kind and not as a collection of determinate triangles. It is an idea, a sign which governs and produces its particular expressions. And which properties were essential to a "general triangle", depends on context, on the activity, and its goals. Even the diagrams of Euclid could be interpreted in two different ways. Under the common interpretation, a general proposition to be proved refers to a definite totality and it says something about each one of its members. Under the other interpretation, no such totality is supposed and the sentence has much more conditional character. Mathematics becomes essentially hypothetic–deductive reasoning or "the science which draws necessary conclusions" (Peirce, 1958, CP 3.558).

When I have an intellectual intuition of an object, Schelling says, "I do not *explain* it, and I do not *deduce* it, but I *contemplate* it. To explain an object is to show how other objects act on it and cause it to act as it does; to deduce an object is to derive it from higher principles, showing how it is only an instance of a general universal" (Beiser, 2008, p. 580).

German Romantic *Naturphilosophie* including Schelling's development of Kant's "dynamism" as opposed to the positivism of the Enlightenment, supported very much exploratory work in fields not reducible to mechanics, like electricity and thermodynamics, and therefore requiring new theoretical ideas. The discovery of the conservation of energy, the most important theorem of the Industrial Revolution—Peirce has called it even "the greatest discovery science has ever made" (CP 6.316)—by Robert Mayer (1815–1878), for instance, owes something to Romantic *Naturphilosophie.* Robert Mayer, by purely speculative reasoning, became convinced of the impossibility of a *perpetuum mobile* and he started one of his essays with a statement of the general principle, that comes down to us from Parmenides' idea of Being as the only thing that is, such that "Is" could not have "come into being", because "nothing comes from nothing". From such considerations Robert Mayer deduced his fundamental discovery, the law of energy conservation.

All phenomena, Robert Mayer argues, "can be derived from an original *Kraft* that works toward overcoming existing differences with the goal of unifying all being into a homogeneous mass at one mathematical point" (Mayer, 1911, p. 1). This would, however, be possible only if the forces that are responsible for the unification could disappear after the task has been accomplished. This does not happen and the basic law of conservation of energy, that is, the basic principle that energies once given are quantitatively unchangeable like matter follows, according to Mayer, ensuring the continued existence of differences, and thus the continued existence of the material world. A general idea, like energy, might therefore be conceived of as the generator of all its concrete manifestations in roughly the same way in which a metaphor unifies the different.

By juxtaposition of two different entities, like in metaphor, one might create something new and more general, that might not be directly observable and of which these entities, A and B, become particular forms or species. Energy is not of interest per se, but its importance is due to the fact that it can take on so many different forms. Robert Mayer had played around a lot with formal equations as well as the analogies suggested by them. Mayer used analogies from chemistry in his reflections on the nature of A=B. And he interpreted such an equation as signifying the transformation of two different forms of the same genus, which he called cause and effect of that transformation, into one another: if the cause is a force the effect must also be a force whatever its form, if the cause is matter the effect must be matter also, etc.

Mayer (1911) understood very well that observation of diversity alone is not sufficient. In an equation, there is always something equal as well as something different. Thus, one had to get the idea that different things might be different forms or appearances of one and the same subject matter. And in order to get that idea one had to experience a multitude of different transformations. Equations were tried out on everything by the proponents of the holistic world view of German *Naturphilosophie*.

Everything must have a meaning, it was believed, and the meaning becomes visible in the process of translation from one representation into another. The idea of continuity ruled the day. Hegel and Schelling had for their "chief topic" (Peirce, 1958) the importance of continuity and continuous evolution. Post-Hegelian evolutionary realism (Peirce) shared the view of the continuum as representing the inexhaustibility of objective reality, but emphasized the interaction of the continuous and the contingent, of *Synechism* and *Tychism* (to use Peirce's terms) in development.

Now the question arises, what these considerations and developments might mean to education and to mathematical education in particular.

According to the ideas of Wilhelm von Humboldt (1767–1835) and his philosophical companions, the contents of instruction were no longer "objects" or particular facts in the treatment of which useful skills and abilities were to be learnt as according to the pedagogy of the Enlightenment. Here for the first time, a theoretical conception of knowledge displaced knowledge in an immediate practical sense. Instead of being oriented to the "needs of daily life"—as Humboldt (1809/1980) described the immediate and pragmatic orientation to societyan orientation was established towards knowledge on the highest level of a theoretical generalization. Astonishingly, at the same time, this caused a radical focus on the individual, more precisely, on that activity, which allows him to realize himself as the subject of his learning. Humboldt expressed this in the following manner:

> With reference to the contents of instruction, from which all original creative work must always follow, the young person should be made capable of already actually beginning to compile the subject matter to a certain extent and to a further extent of accumulating it as he pleases in the future and of developing his intellectual powers. Thus, he is occupied in a twofold manner: with learning, but also with learning how to learn. (Humboldt, 1809, p. 169)

Within the scope of the pedagogy of the Enlightenment, “mechanical skills” were developed—particularly with regard to the technical handling of articles for work, their material prerequisites and means. This accounted, for example, for a large part of the instruction that took place in the industrial schools. The suggestion of Humboldt (1809) signalized a fundamental change. Instead of a direct adoption of articles, substances, and knowledge as a finished product, the activity of learning itself became the focus, but not simply as some sort of automatism or goal governed action. Here, the characterization of learning as *simultaneously* being an orientation towards the content “from which all original creative work must always follow” and an orientation towards creativity and something which was called “learning how to learn” as a conscious focus on the learning process itself seems to be of primary importance. The learning process should be self-organized, autotelic, and oriented simultaneously towards its own development as well as focused on certain tasks or products. A theory or a work of art simultaneously represents some knowledge and meta-knowledge.

The actual historical development was, however, not uniform and consistent. Besides the idealism and esthetical romanticism, which followed on Kant and which prepared the bed for the new Humboldtian conception of mathematics, science and education, critically opposing all kinds of empiricism and technological or economical utilitarism, there was another thread of development in direct continuity to Enlightenment and utilitarian empiricism.

Jeremy Bentham (1748–1832), for example, not entirely to his delight, found that it may well be impossible to speak about reality in a sensible manner without invoking a certain irreducible type of fiction. Condillac (1715–1780) had expressed similar views already somewhat earlier (Otte, 2008, p. 66). Fictions, for Bentham (1814), are

> those sorts of objects which in every language must, for the purpose of the discourse, be spoken of as existing…but without any such danger as that of producing any such persuasion as that of their possessing, each for itself, any separate, or strictly speaking, any real existence. (Bentham, 1814, chap. I)

It is to language, says Bentham, that fictitious entities owe their “impossible yet indispensable” existence. They are, on the one hand, the source of confusion and should ideally be banished, but are on the other hand an indispensable tool. The totality of the legal system is such a fiction but “one generally avoids visiting places where this illusion is shattered” (ibid.).

Already in the days of Condillac and Bentham, we can sense the outline of possible strategies for dealing with the difficulty of these necessary fictions. Bentham may well have been the first to attempt to remove these entities from language by classifying them initially, as fictions of the first degree, second degree, and so on. Bentham’s strategy is based on the idea that all meaningful sentences of a language are reducible to logical constructs upon immediate experiences. If all metaphorical expressions were banned from a linguistic system, however, it would serve to enunciate exclusively that which has been determined by the system’s conventions and all messages would be tautological.

A revival of Bentham’s approach became known in the twentieth century as logical empiricism or logical positivism. In addition to other difficulties of this doctrine, it seems to leave mathematics in a problematic position. Mathematics was deprived of its content and became transformed into an immense tautology; it was considered a language whose meaning is not in its relationship with reality but is somehow self-contained and discovered by suitable methods. But neither was the real meaning or significance of this formal approach completely clear, nor were its effects uniform.

Besides the rigor movement of arithmetization, foremost concerned with ontology, or, in semiotic terms, with the reference relation of mathematical symbols, another quite different epistemological approach, dedicated to more or less radical conceptual generalization and based on a way of analytical thinking in terms of structural analogies, presented itself, which had its roots in Grassmann's work and culminated in Hilbert's new axiomatics as presented in his "Grundlagen der Geometrie" (Foundations of Geometry) of 1899 and in the conception of modern algebra as developed by Emmy Noether and her school (Hilbert, 1977).

Philosophically the complementarity of these two approaches was not readily recognized because of the strict subject–object dichotomy. Or stated in semiotic terms: it was not recognized that the object of a sign and its interpretant are not as different as might appear at first.

5 From the subject–object dichotomy towards Thirdness

Theories are signs and signs must be accepted as realities *sui generis* and at the same time in reference to their objects, that is, as representations. Concepts or theories have to be accepted as real before we can ask for their meaning. This was the problem that concerned Humboldt in the nineteenth century and Thom, Bruner, Davidov, and others about 150 years after Humboldt. Artistic and scientific knowledge is always formal and requires the renunciation of an immediate reification. Theories or works of art are not a doubling or copy of reality but are some kind of fictions in the sense of Bentham. Knowledge is given as a form. The notion of "form" indicates a difference of something from the rest of the world, as it occurs when we try to concentrate on it. It means a distinction, like when Spencer-Brown (1979) begins his "Laws of Form" by requesting: "Draw a distinction!" All mathematics and exact science is formal in this sense, rather than being formal in the sense of mechanical and automated. Form is not the same thing as formalism.

As representing does not mean copying, philosophers can write books about "How the Laws of Physics lie" (Cartwright, 1983). Theories must be simpler than the realities they are supposed to organize. And "there is no reason to think that the principles that best organize will be true, nor that the principles that are true will organize much" (Cartwright, 1983, p. 53).

But theories may also lie like fictions. Like fictions, they largely grow on the basis of an intensional semantics and by intrinsic motives. Theories relate to theories and pictures to pictures in the first place. Works of art or theories are first of all to be understood as elements of a structural context, of a style, or as a certain type of activity and only mediated by this intrinsic understanding may we learn from them something for "real life" by applying the new distinctions and perspectives acquired. "A Constable painting of Marlborough Castle is more like any other picture than it is like the Castle, yet it represents the Castle and not another picture" (Goodman, 1976, p. 5). The painting makes us see the Castle in a new way, rather than being merely an instance in a chain of conventional symbols.

A mathematical text or a work of art is, on the one hand, an object, that is, it is governed by strict determination and necessity, such that not a single word or formula might be changed and, on the other hand, it is a representation and as such it is open to a quasi unlimited range of interpretations. And it seems as if by means of the theory or the work of art as a form, as a reality in its own right, a new freedom would arise, as if new options of seeing would be opened up, which in the end are, however, of a primarily metaphorical character. There is never a literal application of a theory. Having interpreted a theory or an artistic product might change transitorily or permanently our way of seeing the world and of acting within it. Thus texts and theories are means and objects at the same time. Means and objects are fully differentiable by their respective moments on individual cognitive activity, but they play a completely symmetric part in the development of cognition. This complementarity (difference and unity) of objects and means accounts for the emergence and dynamism of pure mathematics in the nineteenth century.

Now, we claim that this complementarity might best be understood from the point of view of cognition as semiotic activity, because semiotic activity is constructive and reflexive at the same time. Texts that I have written in the past allow, for example, an interaction between my yesterday ego and me today. Semiotic activity is goal oriented and simultaneously self-organized or autotelic, because mastery or accomplishment presupposes that "one has the habit of thinking and combining directly from the means, of imagining a work only within the limits of the means at hand, and never approaching a work from a topic or an imagined effect that is not linked to the means" as Paul Valery says (see Otte, 2008, p. 71ff.).

The systematic basis for self-organization is obviously that there is outside of an emerging order no useful or superior information regarding the constructive options and order requirements. There is for example in our society no institution which could incontestably prescribe knowledge claims superior to scientific expertise. The opposite of self-organization is then that of teleology, precisely because teleology means the case in which there is an external institution, which possesses such superior information.

Take the example of painting. A picture is certainly a teleological product because it is a realization of somebody's imagination. But it has also shows highly self-organizing properties. Creating a work of art is not just the mechanical execution according to some pre-conceived idea or prefabricated master plan. In the process of its realization occurs an increasing sensitivity to ideological errors, false intuitions, and other interference of the artist. The wrong color, an ill-chosen direction of the brush, a bad overall distribution of colors and figures, an inattentively outlined figure can jeopardize the achieved order structure at any time, so that in the process of painting the controversy between the direct implementation of an idea, on the one hand, and the casual and contingent distribution of lines or colors, in the manner of Jackson Pollock or the Surrealists, for example, on the other hand, is increasingly replaced by the experience that only the image itself will provide the necessary clues of how to continue and complete the work at hand. It is the picture rather than the painter or the content represented which rules over the creative process. Direct endeavor, says Peirce, "can achieve almost nothing" (Peirce, CP 6.301) and the same is true with respect to mere accidental or casual behavior.

So thinking, learning, painting, etc., all is essentially a semiotic activity which is oriented, developed, and carried to higher levels by the logic and coherence of its own products. Peirce employs the very same analogy to describe the creative process:

> The work of the poet or novelist is not so utterly different from that of the scientific man. The artist introduces a fiction; but it is not an arbitrary one; it exhibits affinities to which the mind accords a certain approval in pronouncing them beautiful, which if it is not exactly the same as saying that the synthesis is true, is something of the same general kind. The geometer draws a diagram, which if not exactly a fiction, is at least a creation, and by means of observation of that diagram he is able to synthesize and show relations between elements which before seemed to have no necessary connection. The realities compel us to put some things into very close relation and

> others less so, in a highly complicated, and in the [to?] sense itself unintelligible manner; but it is the genius of the mind, that takes up all these hints of sense, adds immensely to them, makes them precise, and shows them in intelligible form in the intuitions of space and time. Intuition is the regarding of the abstract in a concrete form by the realistic hypostatization of relations; that is the one sole method of valuable thought. (Peirce, 1958, CP 1.383)

From these examples, we understand that the human mind is a functional goal-oriented system designed to carry out an immense number of possible tasks. It has an organization by virtue of its evolutionary history. This evolution is, however, not simply biological, but has to be accounted for in socio-cultural terms also. And cultural evolution in turn is to be conceived of essentially as a semiotic process, that is, as semiosis. "Man is a sign", Peirce repeatedly said (CP 5.314).

"Man is a sign" also means that man is not completely free in his choice of perspective, ideas and of words or symbols to communicate ideas, if he wants to be understood. The human subjects, as they have neither the possibility of a direct influence on each other nor on the objective world, appear more as dependent agents of the communication process or other productive practices, than as their sovereign. If I understand a work, I feel first of all the difference it causes in my seeing the world and in a second step I might perceive the activities that produced it. To interpret then means to assume the point of view of the producer. That I understand a work of art or theory can then be seen, by my being able to produce something after the same style or method. Of course everybody lives in his private world and many evil or even creative things done are the result of a disengagement from the social world and the communication, which constitutes it. But no artist, scientist, or author alone determines the meaning of his products. And one should mention in addition that the author's influence depends on his familiarity with the ways a certain group, be they artists or mathematicians, see the world.

In face of an overly complex reality and its random noise, technology and method serve as a first and indispensable compass to scientific or cognitive orientation. Newton had understood this already and therefore he opted for a "methodological" concept of mathematization of natural philosophy, rather than a philosophical, that is, ontological one, like Leibniz (see Lenhard & Otte, 2010). "Hypotheses non fingo!" he had famously said to give an answer to those who had publicly challenged him to give an explanation for the causes of gravity rather than just the mathematical principles of mechanics. Newton was much more interested in discovering causes than in understanding and explaining causation. And Kant held the same conviction and expressed it in his "Copernican Revolution" of epistemology.

But technology and methods are merely functional elements of the theory or the work of art, thus depending on the latter. And from the theory's point of view, we must expect that somehow the question referred to has no clear answer at all, and then logic and method do not help really, but one has to generalize. Peirce writes:

> As a general maxim in scientific method, I maintain that at one stage of inquiry it is quite right to insist strongly on the exactitude of established laws, to question which would only lead to confusion, while at a later stage it is proper to question the exactitude of those same laws when we are in possession of a guiding idea which shows us in what manner they may possibly be corrected…What I propose to do tonight is, following the lead of those mathematicians who question whether the sum of the three angles of a triangle is exactly equal to two right angles, to call in question the perfect accuracy of the fundamental axiom of logic. This axiom is that real things

exist, or in other words, what comes to the same thing, that every intelligible question whatever is susceptible in its own nature of receiving a definite and satisfactory answer, if it be sufficiently investigated by observation and reasoning. … Let me be quite understood. As far as all ordinary and practical questions go I insist on this axiom as much as ever….. (Peirce, 1982, W4, pp. 545–546)

As one of the pertinent examples of this questioning of the ontological premises of common logic Peirce cites the creation of projective geometry as an idea which may be applied to philosophy, because projective geometry is no more about definite objects or figures and their respective properties, but deals with relational structures and their possible transformations. "Painters are accustomed to think of a picture as consisting geometrically of the intersections of its plane by rays of light from the natural objects to the eye. But geometers use a generalized perspective", which is governed by the exigencies of mathematical theory, as a coherent and complete whole (Peirce, 1958, CP 6.26).

Since the beginning of the nineteenth century when pure mathematics in the modern sense arose, mathematicians were trying to construct theories as self-referential coherent wholes and the creation of projective geometry was a very prominent example in that direction. It mattered to Peirce also, not least because the "continuity principle" serves as the main source of generality in projective geometry. The excitement about this case has perhaps been greater in England and the USA with their empiricist traditions in philosophy than on the continent.

Geometry's place in the picture of knowledge was originally bound up with its special truth status due to the fact that the objects of geometrical reasoning were clear and distinct, while in analysis and algebra it was often unclear what the symbols meant (see for example Cauchy's introduction to his *Cours d'Analyse* of 1821). The new geometrical development called this into question, but it caused a lot of excitement as an example of a wholly new approach to mathematical theory and methodology. It held as much interest to the new abstract axiomatic or invariant theoretical approach of Hilbert, Cayley, and others, as it pleased the "modernizers" in the educational camp. Joan Richards, for example, writes, with respect to these modernizers in Victorian England:

Although their primary interests were mathematical, these men were active members of the larger intellectual community of post-Darwinian England….Much of what concerned men like Huxley and Tyndall, or their more mathematical friends like Hirst and Clifford was that science in England needed to be more adequately supported. (Richards, 1988, p. 132)

And in this context, "projective geometry was presented as the new naturalistic study of space…In this way, it could be pursued as an integral part of the progressive scientific vision being propounded by a new generation of post-Darwinian scientific publicists" (Richards, 1988, p. 137).

6 Conclusion

The general lesson for mathematical education could be that the semiotic perspective might be helpful to promote a less empiricist and less reductionist view of mathematical explanation, than that which still prevails in today's classrooms. We should remember Bateson's or Feynman's statements, which have reminded us that the thinking is not in the head but is in the sign, in the painting or the text or whatever. One might have stated that learning and thinking are semiotic activities, which seems correct to say as soon as one understands by the term "activity" not just a process, but conceives of it as a rather complicated system, which first of all establishes and develops the epistemic subject-object relationship (see Radford & Roth, this issue). And neither a simple "learning by doing" nor direct instruction will work really, because it is communication as a reality *sui generis* which is the master of knowledge and not the teacher or the learner. Mind is a sign with distributed embodiments.

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